

EXPONENTIATION OF MOTIVIC MEASURES

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ABSTRACT. In this short note we establish some properties of all those motivic measures which can be exponentiated. As a first application, we show that the rationality of Kapranov's zeta function is stable under products. As a second application, we give an elementary proof of a result of Totaro.

1. MOTIVIC MEASURES

Let k be an arbitrary base field and $\text{Var}(k)$ the category of *varieties*, i.e. reduced separated k -schemes of finite type. The *Grothendieck ring of varieties* $K_0\text{Var}(k)$ is defined as the quotient of the free abelian group on the set of isomorphism classes of varieties $[X]$ by the relations $[X] = [Y] + [X \setminus Y]$, where Y is a closed subvariety of X . The multiplication is induced by the product over $\text{Spec}(k)$. When k is of positive characteristic, one needs also to impose the relation $[X] = [Y]$ for every surjective radicial morphism $X \rightarrow Y$; see Mustață [19, Page 78]. Let $\mathbf{L} := [\mathbb{A}^1]$.

The structure of the Grothendieck ring of varieties is quite mysterious; see Poonen [21] for instance. In order to capture some of its flavor several *motivic measures*, i.e. ring homomorphisms $\mu : K_0\text{Var}(k) \rightarrow R$, have been built. Examples include the counting measure $\mu_{\#}$ (see [19, Ex. 7.7]); the Euler characteristic measure χ_c (see [19, Ex. 7.8]); the Hodge characteristic measure μ_H (see [14, §4.1]); the Poincaré characteristic measure μ_P with values in $\mathbb{Z}[u]$ (see [14, §4.1]); the Larsen-Lunts “exotic” measure μ_{LL} (see [13]); the Albanese measure μ_{Alb} with values in the semigroup ring of isogeny classes of abelian varieties (see [19, Thm. 7.21]); the Gillet-Soulé measure μ_{GS} with values in the Grothendieck ring $K_0(\text{Chow}(k)_{\mathbb{Q}})$ of Chow motives (see [6]); and the measure μ_{NC} with values in the Grothendieck ring of noncommutative Chow motives (see [23]). There exist several relations between the above motivic measures. For example, $\chi_c, \mu_H, \mu_P, \mu_{\text{NC}}$, factor through μ_{GS} .

2. KAPRANOV'S ZETA FUNCTION

As explained in [19, Prop. 7.27], in the construction of the Grothendieck ring of varieties we can restrict ourselves to quasi-projective varieties. Given a motivic measure μ , Kapranov introduced in [11] the associated zeta function

$$(2.0.1) \quad \zeta_{\mu}(X; t) := \sum_{n=0}^{\infty} \mu([S^n(X)]) t^n \in (1 + R[[t]])^{\times},$$

where $S^n(X)$ stands for the n^{th} symmetric product of the quasi-projective variety X . In the particular case of the counting measure, (2.0.1) agrees with the classical Weil zeta function. Here are some other computations (with X smooth projective)

$$\zeta_{\chi_c}(X; t) = (1 - t)^{-\chi_c(X)} \quad \zeta_P(X; t) = \prod_{r \geq 0} \left(\frac{1}{1 - u^r t} \right)^{(-1)^{b_r}} \quad \zeta_{\text{Alb}}(X; t) = \frac{[\text{Alb}(X)]t}{1 - t},$$

where $b_r := \dim_{\mathbb{C}} H_{dR}^r(X)$ and $\text{Alb}(X)$ is the Albanese variety of X ; see [22, §3].

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3. BIG WITT RING

Given a commutative ring R , recall from Bloch [2, Page 192] the construction of the big Witt ring $W(R)$. As an additive group, $W(R)$ is equal to $((1 + R[[t]])^\times, \times)$. Let us write $+_W$ for the addition in $W(R)$ and $1 = 1 + 0t + \dots$ for the zero element. The multiplication $*$ in $W(R)$ is uniquely determined by the following requirements:

- (i) The equality $(1 - at)^{-1} * (1 - bt)^{-1} = (1 - abt)^{-1}$ holds for every $a, b \in R$;
- (ii) The assignment $R \mapsto W(R)$ is an endofunctor of commutative rings.

The unit element is $(1 - t)^{-1}$. We have also a (multiplicative) Teichmüller map

$$R \longrightarrow W(R) \quad a \mapsto [a] := (1 - at)^{-1}$$

such that $g(t) * [a] = g(at)$ for every $a \in R$ and $g(t) \in W(R)$; see [2, Page 193].

Definition 3.1. Elements of the form $p(t) -_W q(t) \in W(R)$, with $p(t), q(t) \in R[t]$ and $p(0) = q(0) = 1 \in R$, are called *rational functions*.

Let $W_{\text{rat}}(R)$ be the subset of rational elements. As proved by Naumann in [20, Prop. 6], $W_{\text{rat}}(R)$ is a subring of $W(R)$. Moreover, $R \mapsto W_{\text{rat}}(R)$ is an endofunctor of commutative rings. Recall also the construction of the commutative ring $\Lambda(R)$. As an additive group, $\Lambda(R)$ is equal to $W(R)$. The multiplication is uniquely determined by the requirement that the involution group isomorphism $\iota : \Lambda(R) \rightarrow W(R), g(t) \mapsto g(-t)^{-1}$, is a ring isomorphism. The unit element is $1 + t$.

4. EXPONENTIATION

Let μ be a motivic measure. As explained by Mustaa in [19, Prop. 7.28], the assignment $X \mapsto \zeta_\mu(X; t)$ gives rise to a group homomorphism

$$(4.0.2) \quad \zeta_\mu(-; t) : K_0 \text{Var}(k) \longrightarrow W(R).$$

Definition 4.1. ([22, §3]) A motivic measure μ can be *exponentiated*¹ if the above group homomorphism (4.0.2) is a ring homomorphism.

Corollary 4.2. *Given a motivic measure μ as in Definition 4.1, the following holds:*

- (i) *The ring homomorphism (4.0.2) is a new motivic measure;*
- (ii) *Any motivic measure which factors through μ can also be exponentiated.*

This class of motivic measures is well-behaved with respect with rationality:

Proposition 4.3. *Let μ be a motivic measure as in Definition 4.1. If $\zeta_\mu(X; t)$ and $\zeta_\mu(Y; t)$ are rational functions, then $\zeta_\mu(X \times Y; t)$ is also a rational function.*

Proof. It follows automatically from the fact that $W_{\text{rat}}(R)$ is a subring of $W(R)$. \square

As proved by Naumann in [20, Prop. 8] (see also [22, Thm. 2.1]), the counting measure $\mu_\#$ can be exponentiated. On the other hand, Larsen-Lunts “exotic” measure μ_{LL} cannot be exponentiated! This would imply, in particular, that

$$(4.0.3) \quad \zeta_{\mu_{\text{LL}}}(C_1 \times C_2; t) = \zeta_{\mu_{\text{LL}}}(C_1; t) * \zeta_{\mu_{\text{LL}}}(C_2; t)$$

for any two smooth projective curves C_1 and C_2 . As proved by Kapranov in [11] (see also [19, Thm. 7.33]), $\zeta_\mu(C; t)$ is a rational function for every smooth projective

¹Note that Kapranov’s zeta function is similar to the exponential function $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. The product X^n corresponds to x^n and the symmetric product $S^n(X)$ corresponds to $\frac{x^n}{n!}$ since $n!$ is the size of the symmetric group on n letters.

curve C and motivic measure μ . Using Proposition 4.3, this hence implies that the right-hand side of (4.0.3) is also a rational function. On the other hand, as proved by Larsen-Lunts in [13, Thm. 7.6], the left-hand side of (4.0.3) is not a rational function whenever C_1 and C_2 have positive genus. We hence obtain a contradiction.

At this point, it is natural to ask which motivic measures can be exponentiated? We now provide a general answer to this question using the notion of λ -ring. Recall that a λ -ring R consists of a commutative ring equipped with a sequence of maps $\lambda^n : A \rightarrow A, n \geq 0$, such that $\lambda^0(a) = 1$, $\lambda^1(a) = a$, and $\lambda^n(a+b) = \sum_{i+j=n} \lambda^i(a)\lambda^j(b)$ for every $a, b \in R$. In other words, the map

$$\lambda_t : R \longrightarrow \Lambda(R) \quad a \mapsto \lambda_t(a) := \sum_n \lambda^n(a)t^n$$

is a group homomorphism. Equivalently, the composed map

$$(4.0.4) \quad \sigma_t : R \xrightarrow{\lambda_t} \Lambda(R) \xrightarrow{\iota} W(R) \quad a \mapsto \sigma_t(a) := \lambda_{-t}(a)^{-1}$$

is a group homomorphism. This homomorphism is called the *opposite* λ -structure.

Proposition 4.4. *Let μ be a motivic measure and R a λ -ring such that:*

- (i) *The above group homomorphism (4.0.4) is a ring homomorphism;*
- (ii) *We have $\mu([S^n(X)]) = \sigma^n(\mu([X]))$ for every quasi-projective variety X .*

Under these conditions, the motivic measure μ can be exponentiated.

Proof. Consider the following composed ring homomorphism

$$(4.0.5) \quad K_0 \text{Var}(k) \xrightarrow{\mu} R \xrightarrow{\sigma_t} W(R).$$

The equalities $\mu([S^n(X)]) = \sigma^n(\mu([X]))$ allow us to conclude that (4.0.5) agrees with the group homomorphism $\zeta_\mu(-; t)$. This achieves the proof. \square

Remark 4.5. Let \mathcal{C} be a \mathbb{Q} -linear additive idempotent complete symmetric monoidal category. As proved by Heinloth in [9, Lem. 4.1], the exterior powers give rise to a special λ -structure on the Grothendieck ring $K_0(\mathcal{C})$, with opposite λ -structure given by the symmetric powers Sym^n . In this case, (4.0.4) is a ring homomorphism.

Remark 4.6. Let \mathcal{T}' be a \mathbb{Q} -linear thick triangulated monoidal subcategory of compact objects in the homotopy category $\mathcal{T} = \text{Ho}(\mathcal{C})$ of a simplicial symmetric monoidal model category \mathcal{C} . As proved by Guletskii in [8, Thm. 1], the exterior powers give rise to a special λ -structure on $K_0(\mathcal{T}')$, with opposite λ -structure given by the symmetric powers Sym^n . In the case, (4.0.4) is a ring homomorphism.

Remark 4.7. Assume that k is of characteristic zero. Thanks to Heinloth's presentation of the Grothendieck group of varieties (see [10, Thm. 3.1]), it suffices to verify the equality $\mu([S^n(X)]) = \sigma^n(\mu([X]))$ for every smooth projective variety X .

As an application of the above Proposition 4.4, we obtain the following result:

Proposition 4.8. *The Gillet-Soulé motivic measure μ_{GS} can be exponentiated.*

Proof. Recall from [6] that μ_{GS} is induced by the symmetric monoidal functor

$$(4.0.6) \quad \mathfrak{h} : \text{SmProj}(k) \longrightarrow \text{Chow}(k)_{\mathbb{Q}}$$

from the category of smooth projective varieties to the category of Chow motives. Since the latter category is \mathbb{Q} -linear, additive, idempotent complete, and symmetric monoidal, Remark 4.5 implies that the Grothendieck ring $K_0(\text{Chow}(k)_{\mathbb{Q}})$ satisfies

condition (i) of Proposition 4.4. As proved by del Baño-Aznar in [4, Cor. 2.4], we have $\mathfrak{h}(S^n(X)) \simeq \text{Sym}^n \mathfrak{h}(X)$ for every smooth projective variety X . Using Remark 4.7, this hence implies that condition (ii) of Proposition 4.4 is also satisfied. \square

Remark 4.9. Thanks to Corollary 4.2(ii), all the motivic measures which factor through μ_{GS} (e.g. $\chi_c, \mu_H, \mu_P, \mu_{\text{NC}}$) can also be exponentiated.

5. APPLICATION I: RATIONALITY OF ZETA FUNCTIONS

By combining Propositions 4.3 and 4.8, we obtain the following result:

Corollary 5.1. *Let X, Y be two varieties. If $\zeta_{\mu_{\text{GS}}}(X; t)$ and $\zeta_{\mu_{\text{GS}}}(Y; t)$ are rational functions, then $\zeta_{\mu_{\text{GS}}}(X \times Y; t)$ is also a rational function.*

Remark 5.2. Corollary 5.1 was independently obtained by Heinloth [9, Prop. 6.1] in the particular case of smooth projective varieties and under the extra assumption that $\zeta_{\mu_{\text{GS}}}(X; t)$ and $\zeta_{\mu_{\text{GS}}}(Y; t)$ satisfy a certain functional equation.

Example 5.3. Let X, Y be smooth projective varieties (e.g. abelian varieties) for which $\mathfrak{h}(X), \mathfrak{h}(Y)$ are Kimura-finite; see [12, §3]. Consider the ring homomorphism

$$(5.0.7) \quad \sigma_t : K_0(\text{Chow}(k)_{\mathbb{Q}}) \longrightarrow W(K_0(\text{Chow}(k)_{\mathbb{Q}})).$$

As proved by André in [1, Prop. 4.6], $\sigma_t([\mathfrak{h}(X)])$ and $\sigma_t([\mathfrak{h}(Y)])$ are rational functions. Since $\zeta_{\mu_{\text{GS}}}(-; t)$ agrees with the composition of μ_{GS} with (5.0.7), these latter functions are equal to $\zeta_{\mu_{\text{GS}}}(X; t)$ and $\zeta_{\mu_{\text{GS}}}(Y; t)$, respectively. Using Corollary 5.1, we hence conclude that $\zeta_{\mu_{\text{GS}}}(X \times Y; t)$ is also a rational function.

Recall from Voevodsky [24, §2.2] the construction of the functor

$$(5.0.8) \quad M^c : \text{Var}(k)^p \longrightarrow \text{DM}_{\text{gm}}(k)_{\mathbb{Q}}$$

from the category of varieties and proper morphisms to the triangulated category of geometric motives. As proved in [24, Prop. 4.1.7], the functor (5.0.8) is symmetric monoidal. Moreover, given a variety X and a closed subvariety $Y \subset X$, we have

$$M^c(Y) \longrightarrow M^c(X) \longrightarrow M^c(X \setminus Y) \longrightarrow M^c(Y)[1];$$

see [24, Prop. 4.1.5]. Consequently, we obtain the following motivic measure

$$(5.0.9) \quad K_0 \text{Var}(k) \longrightarrow K_0(\text{DM}_{\text{gm}}(k)_{\mathbb{Q}}) \quad [X] \mapsto [M^c(X)].$$

Proposition 5.4. *The above motivic measure (5.0.9) agrees with μ_{GS} .*

Proof. As proved by Voevodsky in [24, Prop. 2.1.4], there exists a \mathbb{Q} -linear additive fully-faithful symmetric monoidal functor

$$(5.0.10) \quad \text{Chow}(k)_{\mathbb{Q}} \longrightarrow \text{DM}_{\text{gm}}(k)_{\mathbb{Q}}$$

such that (5.0.10) $\circ \mathfrak{h}(X) \simeq M^c(X)$ for every smooth projective variety. Thanks to the work of Bondarko [3, Cor. 6.4.3 and Rk. 6.4.4], the above functor (5.0.10) induces a ring isomorphism $K_0(\text{Chow}(k)_{\mathbb{Q}}) \simeq K_0(\text{DM}_{\text{gm}}(k)_{\mathbb{Q}})$. Therefore, the proof follows from Heinloth's presentation of the Grothendieck ring of varieties in terms of smooth projective varieties; see [10, Thm. 3.1]. \square

Thanks to Proposition 5.4, Example 5.3 admits the following generalization:

Example 5.5. Let X, Y be varieties for which $M^c(X), M^c(Y)$ are Kimura-finite. Similarly to Example 5.3, $\zeta_{\mu_{\text{GS}}}(X \times Y; t)$ is then a rational function.

In the above Examples 5.3 and 5.5, the rationality of $\zeta_{\mu_{\text{GS}}}(X \times Y; t)$ can alternatively be deduced from the stability of Kimura-finiteness under tensor products; see [12, §5]. Thanks to the work of O’Sullivan-Mazza [18, §5.1] and Guletskii [8], the above Corollary 5.1 can also be applied to non Kimura-finite situations.

Proposition 5.6. *Let X_0 be a connected smooth projective surface over an algebraically closed field k_0 such that $q = 0$ and $p_g > 0$, $k := k_0(X_0)$ the function field of X_0 , x_0 a k_0 -point of X_0 , z the zero-cycle which is the pull-back of the cycle $\Delta(X_0) - (x_0 \times X)$ along $X_0 \times k \rightarrow X_0 \times X_0$, Z the support of z , and finally U the complement of Z in $X = X_0 \times k$. Under these notations, the following holds:*

- (i) *The geometric motive $M^c(U)$ is not Kimura-finite;*
- (ii) *Kapranov’s zeta function $\zeta_{\mu_{\text{GS}}}(U; t)$ is rational.*

Proof. As proved by O’Sullivan-Mazza in [18, Thm. 5.18], $M(U)$ is not Kimura-finite. Since the surface U is smooth, we have $M^c(U) \simeq M(U)^*(2)[4]$; see [24, Thm. 4.3.7]. Using the fact that $-(2)[4]$ is an auto-equivalence of the category $\text{DM}_{\text{gm}}(k)_{\mathbb{Q}}$ and that $M(U)^*$ is Kimura-finite if and only if $M(U)$ is Kimura-finite (see Deligne [5, Prop. 1.18]), we conclude that $M^c(U)$ also is not Kimura-finite.

We now prove item (ii). As proved by Guletskii in [8, §3], the category $\text{DM}_{\text{gm}}(k)_{\mathbb{Q}}$ satisfies the conditions of Remark 4.6. Consequently, we have a ring homomorphism

$$(5.0.11) \quad \sigma_t : K_0(\text{DM}_{\text{gm}}(k)_{\mathbb{Q}}) \longrightarrow W(K_0(\text{DM}_{\text{gm}}(k)_{\mathbb{Q}})).$$

As explained by Guletskii in [8, Ex. 5], $\sigma_t([M(U)])$ is a rational function. Thanks to Lemma 5.7 below, we hence conclude that $\sigma_t([M^c(U)])$ is also a rational function. The proof follows now from the fact that $\zeta_{\mu_{\text{GS}}}(-; t)$ agrees with the composition of the ring homomorphisms (5.0.9) and (5.0.11). \square

Lemma 5.7. *Given a smooth variety X of dimension d , we have the equality*

$$\sigma_t([M^c(X)]) = \sigma_{\mu_{\text{GS}}(\mathbf{L})^{d_t}}([M(X)]).$$

Proof. The proof is given by the following identifications

$$(5.0.12) \quad \begin{aligned} \sigma_t([M^c(X)]) &= \sigma_t([M(X)^*(d)[2d]]) \\ &= \sigma_t([M(X)^*] \mu_{\text{GS}}(\mathbf{L}^d)) \\ &= \sigma_t([M(X)^*]) * \zeta_{\mu_{\text{GS}}}(\mathbf{L}^d; t) \end{aligned}$$

$$(5.0.13) \quad = \sigma_t([M(X)]) * \zeta_{\mu_{\text{GS}}}(\mathbf{L}^d; t)$$

$$(5.0.14) \quad = \sigma_{\mu_{\text{GS}}(\mathbf{L})^{d_t}}([M(X)]),$$

where (5.0.12) follows from [24, Thm. 4.3.7], (5.0.13) from [5, Lem. 1.18], and (5.0.14) from Remark 6.2 below with $\mu := \mu_{\text{GS}}$ and $g(t) := \sigma_t([M(X)])$. \square

Example 5.8. Let U_1, U_2 be two surfaces as in Proposition 5.6. Thanks to the above Corollary 5.1, we hence conclude that $\zeta_{\mu_{\text{GS}}}(U_1 \times U_2; t)$ is a rational function.

Remark 5.9. Thanks to Corollary 4.2(ii), the above Examples 5.3, 5.5, and 5.8, hold *mutatis mutandis* for any motivic measure which factors through μ_{GS} .

6. APPLICATION II: TOTARO’S RESULT

The following result plays a central role in the study of the zeta functions.

Proposition 6.1 (Totaro). *The equality $\zeta_{\mu}(X \times \mathbb{A}^n; t) = \zeta_{\mu}(X; \mu(\mathbf{L})^n t)$ holds for every variety X and motivic measure μ .*

Its proof (see [7, Lem. 4.4][19, Prop. 7.32]) is non-trivial and based on a stratification of the symmetric products of $X \times \mathbb{A}^n$. In all the cases where the motivic measure μ can be exponentiated, this result admits the following elementary proof:

Proof. Since $[X \times \mathbb{A}^n] = [X][\mathbb{A}^n]$ in the Grothendieck ring of varieties and the motivic measure μ can be exponentiated, the proof is given by the identifications

$$\begin{aligned}
 \zeta_\mu(X \times \mathbb{A}^n; t) &= \zeta_\mu(X; t) * \zeta_\mu(\mathbf{L}^n; t) \\
 &= \zeta_\mu(X; t) * \zeta_\mu(\mathbf{L}; t)^{*n} \\
 (6.0.15) \quad &= \zeta_\mu(X; t) * (1 + \mu(\mathbf{L})t + \mu(\mathbf{L})t^2 + \cdots)^{*n} \\
 &= \zeta_\mu(X; t) * ((1 - \mu(\mathbf{L})t)^{-1})^{\ast n} \\
 &= \zeta_\mu(X; t) * [\mu(\mathbf{L})]^{\ast n} \\
 &= \zeta_\mu(X; t) * [\mu(\mathbf{L})^n] \\
 &= \zeta_\mu(X; \mu(\mathbf{L})^n t),
 \end{aligned}$$

where (6.0.15) follows from [19, Ex. 7.23] and $[\mu(\mathbf{L})]$ stands for the image of $\mu(\mathbf{L}) \in R$ under the multiplicative Teichmüller map $R \rightarrow W(R)$. \square

Remark 6.2. The above proof shows more generally that $g(t) * \zeta_\mu(\mathbf{L}^n; t) = g(\mu(\mathbf{L})^n t)$ for every $g(t) \in W(R)$ and motivic measure μ which can be exponentiated.

Remark 6.3. (Fiber bundles) Given a fiber bundle $E \rightarrow X$ of rank n , we have $[E] = [X][\mathbb{A}^n]$ in the Grothendieck ring of varieties; see [19, Prop. 7.4]. Therefore, the above proof, with X replaced by E , shows that $\zeta_\mu(E; t) = \zeta_\mu(X; \mu(\mathbf{L})^n t)$.

Remark 6.4. (\mathbb{P}^n -bundles) Given a \mathbb{P}^n -bundle $E \rightarrow X$, we have $[E] = [X][\mathbb{P}^n]$ in the Grothendieck ring of varieties; see [19, Ex. 7.5]. Therefore, by combining the equality $[\mathbb{P}^n] = 1 + \mathbf{L} + \cdots + \mathbf{L}^n$ with the above proof, we conclude that

$$\zeta_\mu(E; t) = \zeta_\mu(X; t) +_W \zeta_\mu(X; \mu(\mathbf{L})t) +_W \cdots +_W \zeta_\mu(X; \mu(\mathbf{L})^n t).$$

7. G -VARIETIES

Let G be a finite group and $\text{Var}^G(k)$ the category of G -varieties, i.e. varieties X equipped with a G -action $\lambda : G \times X \rightarrow X$ such that every orbit is contained in an affine open set. The *Grothendieck ring of G -varieties* $K_0\text{Var}^G(k)$ is defined as the quotient of the free abelian group on the set of isomorphism classes of G -varieties $[X, \lambda]$ by the relations $[X, \lambda] = [Y, \tau] + [X \setminus Y, \lambda]$, where (Y, τ) is a closed G -invariant subvariety of (X, λ) . The multiplication is induced by the product of varieties. A motivic measure is a ring homomorphism $\mu^G : K_0\text{Var}^G(k) \rightarrow R$. As mentioned in [15, §5], the above measures χ_c, μ_H, μ_P admit G -extensions $\chi_c^G, \mu_H^G, \mu_P^G$.

Notation 7.1. Let $\text{Chow}^G(k)_\mathbb{Q}$ be the category of functors from the group G (considered as a category with a single object) to the category $\text{Chow}(k)_\mathbb{Q}$.

Note that $\text{Chow}^G(k)_\mathbb{Q}$ is still a \mathbb{Q} -linear additive idempotent complete symmetric monoidal category and that (4.0.6) extends to a symmetric monoidal functor

$$(7.0.16) \quad \mathfrak{h}^G : \text{SmProj}^G(k) \longrightarrow \text{Chow}^G(k)_\mathbb{Q}.$$

Note also that the n^{th} symmetric product of a G -variety is still a G -variety. Therefore, the notion of exponentiation makes sense in this generality. Gillet-Soulé's motivic measure μ_{GS} admits the following G -extension:

Proposition 7.2. *The above functor (7.0.16) gives rise to a motivic measure*

$$\mu_{\text{GS}}^G : K_0 \text{Var}^G(k) \longrightarrow K_0(\text{Chow}^G(k)_{\mathbb{Q}})$$

which can be exponentiated.

Proof. Given a smooth projective variety X and a closed subvariety Y , let us denote by $\text{Bl}_Y(X)$ the blow-up of X along Y and by E the associated exceptional divisor. As proved by Manin in [16, §9], we have a natural isomorphism $\mathfrak{h}(\text{Bl}_Y(X)) \oplus \mathfrak{h}(Y) \simeq \mathfrak{h}(X) \oplus \mathfrak{h}(E)$ in $\text{Chow}(k)_{\mathbb{Q}}$. Since this isomorphism is natural, it also holds in $\text{Chow}^G(k)_{\mathbb{Q}}$ when X is replaced by a smooth projective G -variety (X, λ) and Y by a closed G -invariant subvariety (Y, τ) . Therefore, thanks to Heinloth's presentation of the Grothendieck ring of G -varieties in terms of smooth projective G -varieties (see [10, Lem. 7.1]), the assignment $X \mapsto \mathfrak{h}^G(X)$ gives rise to a (unique) motivic measure μ_{GS}^G . The proof of Proposition 4.8, with (4.0.6) replaced by (7.0.16), shows that this motivic measure μ_{GS}^G can be exponentiated. \square

Remark 7.3. Thanks to Corollary 4.2(ii), all the motivic measures which factor through μ_{GS}^G (e.g. $\chi_c^G, \mu_{\text{H}}^G, \mu_{\text{P}}^G$) can also be exponentiated.

Proposition 4.3 admits the following G -extension:

Proposition 7.4. *Let μ^G be a motivic measure which can be exponentiated and $(X, \lambda), (Y, \tau)$ two G -varieties. If $\zeta_{\mu^G}((X, \lambda); t)$ and $\zeta_{\mu^G}((Y, \tau); t)$ are rational functions, then $\zeta_{\mu^G}((X \times Y, \lambda \times \tau); t)$ is also a rational function.*

Example 7.5. Assume that the group G (of order r) is abelian and that the base field k is algebraically closed of characteristic zero or of positive characteristic p with $p \nmid r$. Under these assumptions, Mazur proved in [17, Thm. 1.1] that $\zeta_{\mu^G}((C, \lambda); t)$ is a rational function for every smooth projective G -curve (C, λ) and motivic measure μ^G . Thanks to Proposition 7.4, we hence conclude that $\zeta_{\mu^G}((C_1 \times C_2, \lambda_1 \times \lambda_2); t)$ is still a rational function for every motivic measure μ^G which can be exponentiated and for any two smooth projective G -curves (C_1, λ_1) and (C_2, λ_2) .

Finally, Totaro's result admits the following G -extension:

Proposition 7.6. *Let μ^G be a motivic measure which can be exponentiated and $(X, \lambda), (\mathbb{A}^n, \tau)$ two G -varieties. When G (of order r) is abelian and k is algebraically closed, Kapranov's zeta function $\zeta_{\mu^G}((X \times \mathbb{A}^n, \lambda \times \tau); t)$ agrees with*

$$\zeta_{\mu^G}((X, \lambda); \mu^G(S^r(\mathbb{A}^n, \tau))t) +_W \zeta_{\mu^G}((X, \lambda); t) * \left(\sum_{l=0}^{r-1} \prod_{i=1}^n \mu^G([\mathbb{A}^1, \tau_i] \cdots [\mathbb{A}^1, \tau_i^l]) t^l \right),$$

where $[\mathbb{A}^n, \tau] = [\mathbb{A}^1, \tau_1] \cdots [\mathbb{A}^1, \tau_n]$.

Proof. Since $[X \times \mathbb{A}^n, \lambda \times \tau] = [X, \lambda][\mathbb{A}^n, \tau]$ in the Grothendieck ring of G -varieties and the motivic measure μ^G can be exponentiated, we have the equality

$$\zeta_{\mu^G}((X \times \mathbb{A}^n, \lambda \times \tau); t) = \zeta_{\mu^G}((X, \lambda); t) * \zeta_{\mu^G}((\mathbb{A}^n, \tau); t).$$

Moreover, as explained in [17, Page 1338], we have the following computation

$$\zeta_{\mu^G}((\mathbb{A}^n, \tau); t) = \frac{1}{1 - \mu^G(S^r(\mathbb{A}^n, \tau))t} \left(\sum_{l=0}^{r-1} \prod_{i=1}^n \mu^G([\mathbb{A}^1, \tau_i] \cdots [\mathbb{A}^1, \tau_i^l]) t^l \right).$$

Therefore, since $(1 - \mu^G(S^r(\mathbb{A}^n, \tau))t)^{-1}$ is the Teichmüller class $[\mu^G(S^r(\mathbb{A}^n, \tau))]$, the proof follows from the combination of the above equalities. \square

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